Lezione 8

$T = \mathbb{R}^2 / \underset{\Gamma}{\underset{\shortparallel}{\mathbb{Z}^2}}$ $\qquad \Gamma = \mathbb{Z}^2 < \mathbb{R}^2 < \mathrm{Diffeo}^+(\mathbb{R}^2)$

$$\mathrm{Diffeo}^+(\mathbb{R}^2)$$
$$\vee$$
$$N(\Gamma) = \{ g \mid g\Gamma = \Gamma g \}$$
$$\shortparallel$$
$$= \{ g \mid g^{-1}\Gamma g = \Gamma \}$$

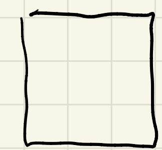
$$\mathrm{Isom}^+(T) = N(\Gamma)/\Gamma$$

$$\mathbb{R}^2 < \mathrm{Diffeo}^+(T) > SL(2,\mathbb{Z})$$

$x \longmapsto Ax \qquad A \in SL(2,\mathbb{Z})$

$$A^{-1}\mathbb{Z}^2 A = \mathbb{Z}^2$$

$$A^{-1}(Ax + b) = x + A^{-1}b \quad \mathbb{Z}^2$$



$\Gamma < \mathbb{R}^2$ RETICOLO $\qquad \Gamma = \langle v, w \rangle_{\mathbb{Z}}$

S_g

Farb-Margalit "A primer on the mapping class group"

Prop: $\gamma \subseteq S_g$ c.s.c.

a) Se γ è omot. banale allora $\gamma = \partial D$

b) " " non è " " " " γ è primitivo in $\pi_1(S_g)$

c) " " non è " " " e semplice $\Rightarrow \bar{\gamma}$ è semplice $\forall$ metrica iperbolica su S_g

dim:

Dimo. a) e b) usando la geom. iperbolica

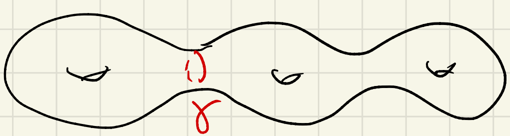
γ omot. banale $\gamma \sim$ cost. $S /\!/ \gamma = S'$ ha 2 c.d. ∂.

a) Se una c.c. di S' è (disco) OK

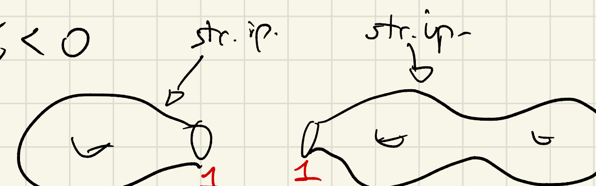
- $S^1 =$  $\rightarrow S = T$ assurdo perché $[\gamma] = 0$

S str. ip.

- S' ha c.c. con $\chi < 0$



$S' = S_1' \cup S_2'$



γ geod. chiusa $\Rightarrow [\gamma] \neq 0$ in $\pi_1(S)$ c

(b) γ è primitiva in $\pi_1(S)$

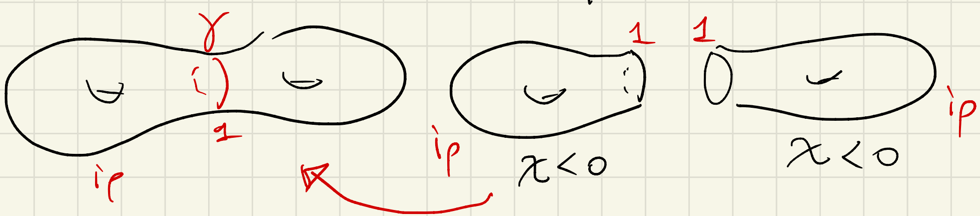
$\pi_1(S) = \Gamma < \mathrm{Isom}(\mathbb{H}^2)$ discreto

p.ass.: $\gamma \sim \eta^k$ $\quad \eta \in \pi_1(S)$ non banale

• non contiene torsione

$S /\!/ \gamma = S'$

Tutte le comp. di S' hanno $\chi < 0$



$\exists$ m.ip. su cui $\gamma = \bar{\gamma}$ è semplice

$\eta \to \bar{\eta}$ c.s.c.

Se p.as. $\gamma = \eta^k$ $\quad \eta^k \to \boxed{(\bar{\eta})^k} = \boxed{\bar{\gamma}}$ ASSURDO

(c) usa (a) e (b)

INTERSEZIONE GEOMETRICA

γ_1, γ_2 c.s.c. in S_g

Def: $i(\gamma_1, \gamma_2) = \min \# \{ \gamma_1' \pitchfork \gamma_2' \mid \gamma_i'$ c.s.c. omotope a $\gamma_i \}$

INTERSEZIONE GEOMETRICA $\geqslant 0$ non è invariante omologico

" " ALGEBRICA : $\sum \pm 1$ dipende solo da $[\gamma_1]\, [\gamma_2]$

$[\gamma_1] \cdot [\gamma_2] = \gamma_1 \cdot \gamma_2$

$i(-\gamma_1, \gamma_2) = i(\gamma_1, \gamma_2)$

$i(\underset{\text{BANALE}}{\gamma}, \eta) = 0$

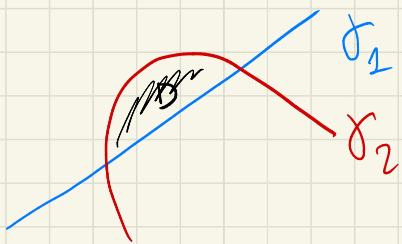
Ex: Nel toro coincidono: $\gamma_1 = (p, q)$ $\gamma_2 = (r, s)$ coprimi

$$|\gamma_1 \cdot \gamma_2| = i(\gamma_1, \gamma_2) = \det \begin{vmatrix} p & r \\ q & s \end{vmatrix}$$

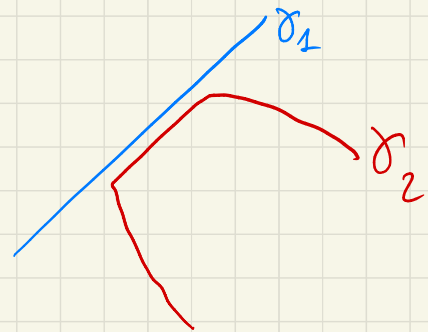
Teo (CRITERIO DEL BIGONO)

$\gamma_1, \gamma_2 \subseteq S$ sono in posiz. minimale trasverse $\Longleftarrow\!\!\Longrightarrow$ non formano bigoni

Def: $\gamma_1, \gamma_2 \subseteq S$ sono in posiz. minimale se $\gamma_1 \pitchfork \gamma_2$ e $\#\{\gamma_1 \cap \gamma_2\} = i(\gamma_1, \gamma_2)$



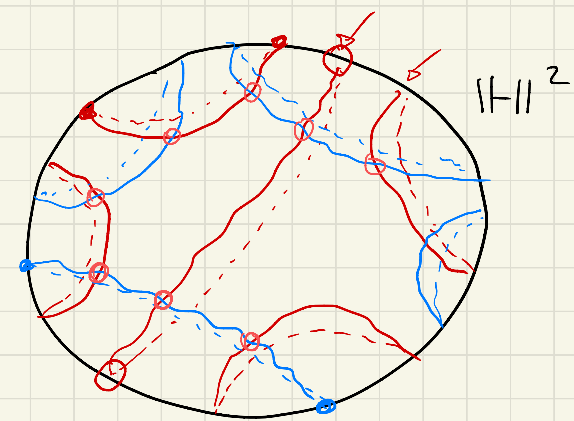
$\boxed{\Longrightarrow}$



$\boxed{\Longleftarrow}$ Metto metrica ip. su S qualsiasi

γ_1, γ_2 non banali

$\bar{\gamma}_1 \quad \bar{\gamma}_2$ rappr. geodetici



$$X = \left\{ (\varphi_1, \varphi_2) \mid \varphi_i \in C(\gamma_i) \text{ t.c. i loro assi sono allacciati} \right\} \Big/ \Gamma$$

$\Big/ \Gamma$

$\updownarrow$

$\{\gamma_1 \cap \gamma_2\}$

$\{\text{rette rosse}\} \longleftrightarrow \{\text{assi di } \varphi \in C(\gamma_1)\}$

$C(\gamma_1) \subseteq \Gamma$ classe di coniugio

$\Gamma = \pi_1(S)$

γ_1', γ_2'

Un arco rosso e un arco blu si intersecano in $\mathbb{H}^2$ in 0/1 punti $\Longleftrightarrow$ lo fanno le rette corrispondenti

0: gli estremi non allacciati

1: " " sono allacciati

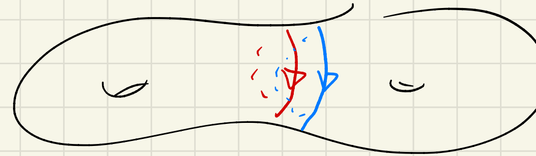
<u>Cor</u>: Due geod. sempl. chiuse distinte sono in posizione minimale □

<u>Prop</u>: Due c.s.c. $\gamma_1, \gamma_2 \subseteq S_g$ omotope $\Longleftrightarrow$ isotope

γ_1 e γ_2 non banali

dim: $\gamma_1 \sim \gamma_2 \Rightarrow i(\gamma_1, \gamma_2) = i(\gamma_1, \gamma_1) = 0$

omot. isot

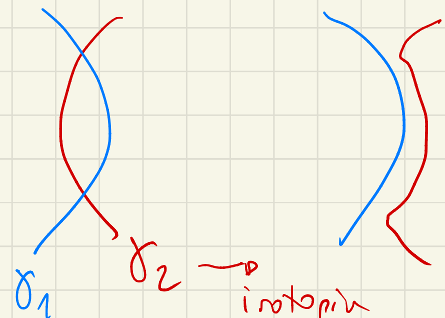


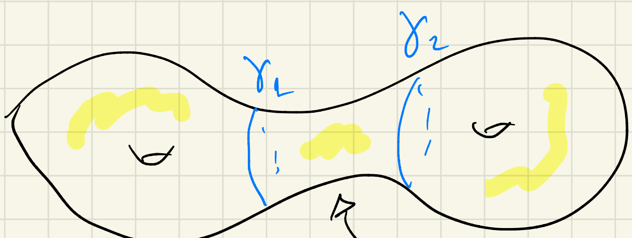
$\gamma_1 \pitchfork \gamma_2$

$\gamma_1 \cap \gamma_2 = \emptyset$

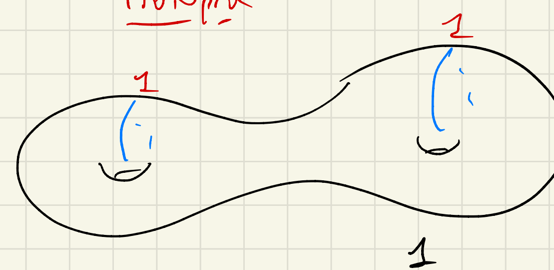
$\gamma_1 \cap \gamma_2 \neq \emptyset \to \exists$ Bigono

non sono minim.





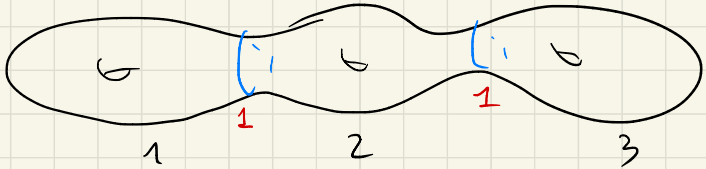
Anello $\to$ isotopia OK



$S /\!/ \gamma_1 \cup \gamma_2$

NO ANELLI NO DISCHI $\to \chi < 0$

$\Rightarrow \exists$ metrica ip. su S in cui γ_1 e γ_2 geod. chiuse disgiunte



$\Rightarrow$ assurdo

$\square$

MAPPING CLASS GROUP

<u>Def</u>: $S = S_{g,b,p}$ $\qquad MCG(S) = \mathrm{Diffeo}^+_\partial(S)\big/_{\text{isot. che fix }\partial} = \{\varphi : S \to S \mid \varphi|_{\partial S} = id_{\partial S}\}\big/_{\text{isot che fix }\partial}$

φ mantiene ori

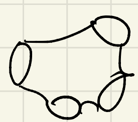
$S = S_g$

$$\boxed{MCG(S)} = \boxed{\mathrm{Diffeo}^+(S)\big/_{\text{isot}}} = \mathrm{Diffeo}^+(S)\big/_{\text{omot}}$$

LIBRO

$$= \boxed{\mathrm{Omeom}^+(S)\big/_{\text{omot}}} = \dots$$

<u>Teo</u> (Smale): $MCG(S^2) = MCG(D^2) = \{e\}$ $\leftarrow$

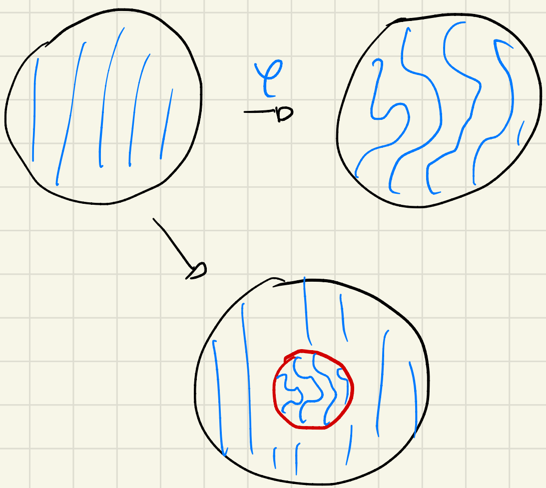


Alexander trick:

$$\boxed{\varphi : D^n \to D^n \text{ omeo} \quad \varphi|_{\partial D^n} = id_{S^{n-1} = \partial D^n}}$$

φ isotopo all'identità continuamente

$$F(x,t) = \begin{cases} x & t \leq \|x\| \\ t\,\varphi\left(\frac{x}{t}\right) & t \geq \|x\| \end{cases}$$

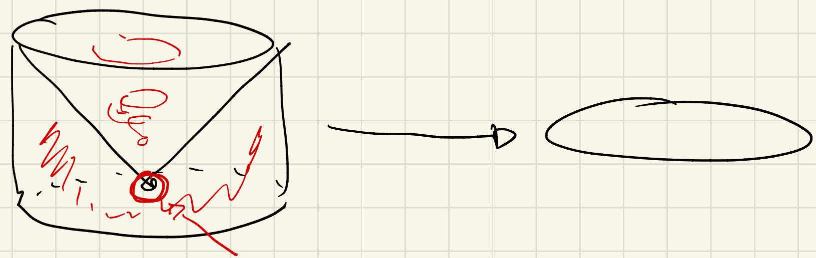


$F : D^n \times [0,1] \to D^n$

$F_1 = f$

$F_0 = id$

F_t omeo $\forall t$



Non è vero nella categoria liscia!

S^7 Milnor

$S = S_g$ $\qquad SL(\mathbb{Z}^{2g}) = SL(2g, \mathbb{Z})$

$$MCG(S) \xrightarrow{\psi} Aut^+(\underbrace{H_1(S,\mathbb{Z})}_{SL(\mathbb{Z}^{2g})})$$

$$\begin{array}{ccc} \psi & & \\ \varphi_1 : S \to S & \dashrightarrow & (\varphi_1)_* : H_1(S) \to H_1(S) \\ \wr & & \parallel \\ \varphi_2 & & (\varphi_2)_* \end{array}$$

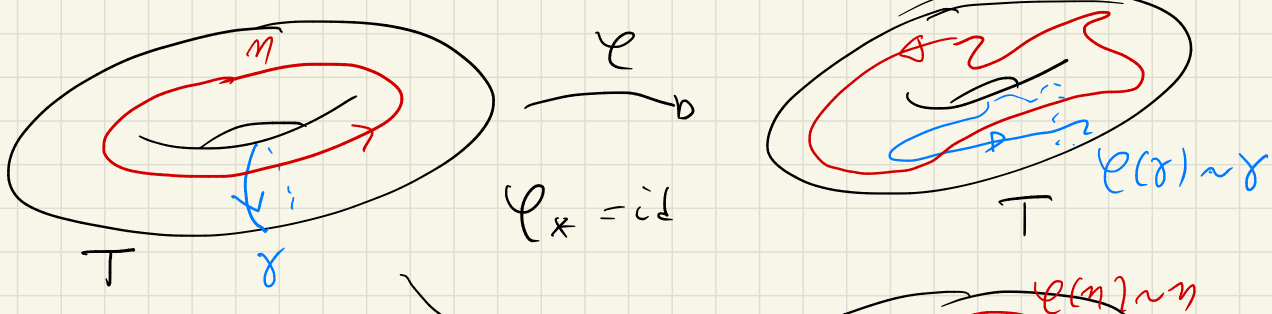
<u>Def</u>: GRUPPO DI TORELLI $:= \ker \psi$

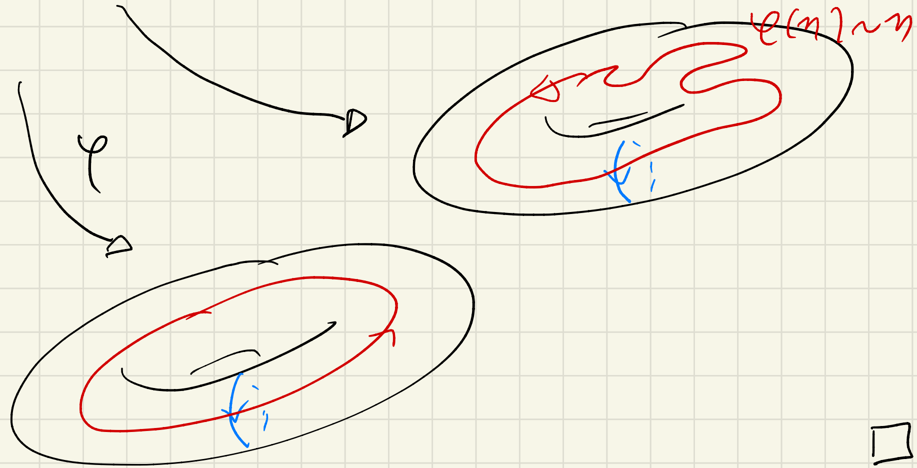
<u>Prop</u>: Per il toro ψ è isom. cioè $MCG(T) \cong SL(2, \mathbb{Z})$

<u>dim</u>:

- Suriettiva: $\forall A \in SL(2,\mathbb{Z}) \quad \exists \ \varphi : T \to T$
- Iniettiva: $\varphi \in Diff^+(T) \qquad \varphi_* = id_{H_1(T)}$

$\overset{?}{\Longrightarrow} \varphi \sim id$





$$T \setminus (\gamma \cup \eta) = \square \cong D^2$$

□